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REPORT TW 56

The influence of an exponential windfield upon a semicircular bay

by

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§ 1. Introduction.

This report is a sequel to TW 47 where Hofsommer considered the free oscillations in a rotating semi-circular bay and to TW 42 and TW 55 in which Lauwerier considered the influence of an exponential windfield upon a rectangular bay. Damsté performed the greater part of the numerical calculations and drew the diagrams.

With the usual approximations the equations of motion and continuity are in Cartesian coordinates x and y

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + \lambda u - \Omega v + c^2 \frac{\partial \zeta}{\partial x} = U \\ &\frac{\partial v}{\partial t} + \lambda v + \Omega u + c^2 \frac{\partial \zeta}{\partial y} = V \\ &\qquad\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \quad , \end{aligned}\right. \tag{1.1}$$

where the symbols have the following meaning

u,v	the components of the total current
ζ	the elevation of the free surface above its equilibrium position which is at $\zeta=0$
Ω	the geostrophic coefficient assumed uniform
λ	a coefficient of friction assumed uniform
h	the undisturbed depth of the water assumed uniform
c	velocity of the free waves $=\sqrt{gh}$ where g is the constant of gravity
U,V	the components of the tractive force of the wind.

The bay covers the area $x^2+y^2 < a^2$, $y < 0$. The semicircle $x^2+y^2=a^2$, $y < 0$ represents a coast, the diameter $y=0$ represents the borderline between the bay and an ocean of infinite depth.

The boundary conditions of 1.1 are

the ocean condition

$$\zeta = 0 \qquad \text{at} \quad y=0, \ -a < x < a, \tag{1.2}$$

and the coast condition

$$\text{radial component of total current} = 0 \tag{1.3}$$

at $x^2+y^2=a^2$, $y < 0$.

We introduce dimensionless quantities according to

$$\begin{cases} x,y = ax',ay' & t = \frac{a}{c}\,t' \\ u,v = hcu',\ hcv' & \zeta = h\,\zeta' \\ U,V = \frac{hc}{a}\,U',\ \frac{hc}{a}\,V' & \text{etc.} \end{cases} \qquad 1.4$$

then with omission of the primes 1.1 becomes

$$\begin{cases} (\frac{\partial}{\partial t} + \lambda)u - \Omega\, v + \frac{\partial \zeta}{\partial x} = U, \\ (\frac{\partial}{\partial t} + \lambda)v + \Omega\, u + \frac{\partial \zeta}{\partial y} = V, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0. \end{cases} \qquad 1.5$$

According to TW 55, formula 3.3, the relation between the dimensionless intensity of wind $\sqrt{U^2+V^2}$ and the windvelocity at sealevel v_s in m/sec is

$$\sqrt{U^2+V^2} = 3 \times 10^{-5}\, v_s^{\,2}\ . \qquad 1.6$$

If next polar coordinates r, φ are introduced by means of

$$x = -r\cos\varphi\ ,\quad y = -r\sin\varphi\ ,\ 0<r<1,\ 0<\varphi<\pi$$

the radial and circular components of the total stream w_r and w_φ are given by

$$\begin{aligned} w_r &= -u\cos\varphi - v\sin\varphi \\ w_\varphi &= u\sin\varphi - v\cos\varphi\ , \end{aligned} \qquad 1.7$$

and similarly for the components W_r, W_φ of the surface stress. By means of this 1.5 reads in polar coordinates

$$\begin{cases} (\frac{\partial}{\partial t} + \lambda)w_r - \Omega\, w_\varphi + \frac{\partial \zeta}{\partial r} = W_r\ , \\ (\frac{\partial}{\partial t} + \lambda)w_\varphi + \Omega\, w_r + \frac{1}{r}\frac{\partial \zeta}{\partial \varphi} = W_\varphi\ , \\ (\frac{\partial}{\partial r} + \frac{1}{r})w_r + \frac{\partial w_\varphi}{\partial \varphi} + \frac{\partial \zeta}{\partial t} = 0\ , \end{cases} \qquad 1.8$$

with the boundary conditions

$$\begin{cases} \zeta = 0 \quad \text{at} \quad \varphi = 0 \ \text{and}\ \varphi=\pi, \\ w_r = 0 \quad \text{at} \quad r = 1\ . \end{cases} \qquad 1.9$$

In the following section we shall determine the Laplace transform of ζ at an arbitrary point of the bay. According to TW 42 this can be used to determine the influence of an arbitrary exponential windfield upon the elevation. The method will be illustrated by solving the following particular numerical case

$h = 65$ meter so that $c = 91$ km/hour
$\Omega = 0.44$ hour^{-1}, $\lambda = 1/5\,\Omega$, $a = c/\Omega = 205$ km.

The value of a is chosen in such a way that the dimensionless values of Ω and λ are respectively 1 and 0.2. The unit of time in 1.8 accordingly is 2.35 hour.
This model represents a bay at a latitude corresponding to that of the North Sea and which has about the same depth and bottom friction.

We shall consider the exponential windfield(dimensionless)

$$\begin{cases} U = 0 \\ V = -e^{0.2t} + 0.2e^{0.3t}. \end{cases} \qquad 1.10$$

The graph of this "northern" wind is given in figure 1. Its duration is of the order of 1½ day.

§2. Solution of the problem

If a Laplace transformation is applied to 1.8 according to

$$\bar{\zeta}(r,\varphi,p) = \int_{-\infty}^{\infty} e^{-pt}\,\zeta(r,\varphi,t)dt, \text{ etc.} \qquad 2.1$$

we obtain

$$\begin{cases} (p+\lambda)\bar{w}_r - \Omega\,\bar{w}_\varphi + \dfrac{\partial\bar{\zeta}}{\partial r} = \bar{W}_r\,, \\ (p+\lambda)\bar{w}_\varphi + \Omega\,\bar{w}_r + \dfrac{1}{r}\dfrac{\partial\bar{\zeta}}{\partial\varphi} = \bar{W}_\varphi, \\ (\dfrac{\partial}{\partial r} + \dfrac{1}{r})\bar{w}_r + \dfrac{\partial\bar{w}_\varphi}{\partial\varphi} + p\bar{\zeta} = 0\,, \end{cases} \qquad 2.2$$

with the boundary conditions

$$\begin{cases} \bar{\zeta} = 0 & \text{at } \varphi = 0 \text{ and } \varphi = \pi\,, \\ \bar{w}_r = 0 & \text{at } r = 1. \end{cases} \qquad 2.3$$

Elimination of $\bar{w}_r$ and $\bar{w}_\varphi$ yields

$$(\Delta - q^2)\bar{\zeta} = \bar{F}\,, \qquad 2.4$$

where Δ is the Laplace operator in polar coordinates,

$$q^2 = p(p+\lambda) + \Omega^2\frac{p}{p+\lambda}\,, \qquad 2.5$$

and

$$\bar{F} = \frac{\partial\bar{U}}{\partial x} + \frac{\partial\bar{V}}{\partial y} + \frac{\Omega}{p+\lambda}(\frac{\partial\bar{V}}{\partial x} - \frac{\partial\bar{U}}{\partial y}) =$$
$$= (\frac{\partial}{\partial r} + \frac{1}{r})\bar{W}_r + \frac{\partial\bar{W}_\varphi}{\partial\varphi} + \frac{\Omega}{p+\lambda}\left\{(\frac{\partial}{\partial r} + \frac{1}{r})\bar{W}_\varphi - \frac{\partial\bar{W}_r}{\partial\varphi}\right\}. \qquad 2.6$$

In terms of $\bar{\zeta}$ the boundary conditions become

$$\bar{\zeta} = 0 \quad \text{at} \quad \varphi = 0 \quad \text{and} \quad \varphi = \pi \ , \qquad 2.7$$

$$\frac{\partial \bar{\zeta}}{\partial r} + \frac{\Omega}{p+\lambda} \frac{\partial \bar{\zeta}}{\partial \varphi} = \overline{W}_r + \frac{\Omega}{p+\lambda} \overline{W}_\varphi \quad \text{at } r = 1 \ . \qquad 2.8$$

If U and v are uniform, their partial derivatives vanish so that $\overline{F}=0$. In this report we consider this case only. The fundamental solutions of 2.4 with $\overline{F}=0$, which satisfy the boundary conditions 2.7 are

$$\bar{\zeta} = I_n(qr) \sin n\varphi \ , \quad n=1,2,3,\ldots \ .$$

We accordingly put

$$\bar{\zeta} = \sum_1^\infty a_n \frac{I_n(qr)}{q\, I_n'(q)} \sin n\varphi \ , \qquad 2.9$$

where the coefficients a_n have to be determined in such a way that the coast condition 2.8 is satisfied. Substitution of 2.9 in 2.8 gives in view of 1.7

$$\sum_1^\infty a_n(\sin n\varphi + \gamma_n \cos n\varphi) = -\overline{U}(\cos\varphi - \gamma \sin\varphi) - \overline{V}(\sin\varphi + \gamma\cos\varphi) \ , \qquad 2.10$$

where

$$\gamma = \frac{\Omega}{p+\lambda} \ , \qquad \gamma_n = \frac{n\, I_n(q)}{q\, I_n'(q)} \gamma \ . \qquad 2.11$$

The quantities θ_n defined by

$$\theta_n = \frac{n\, I_n(q)}{q\, I_n'(q)} \qquad 2.12$$

satisfy the differential equation

$$\frac{d\theta_n}{dq} = \frac{n}{q}(1-\theta_n{}^2) + \frac{q}{n} \ . \qquad 2.13$$

They can be written as a power series in q^2 as follows

$$\theta_n = 1 + \frac{1}{2n(n+1)} q^2 - \frac{1}{8n(n+1)^2(n+2)} q^4 + \ldots \ . \qquad 2.14$$

Hence, for large values of n, $\theta_n = 1 + O(n^{-2})$.

The coefficients a_n in 2.9 are to be determined from 2.10. We write the latter equation in the form

$$\sum_1^\infty a_n(\sin n\varphi + \gamma \cos n\varphi) = g(\varphi) \ , \qquad 2.15$$

$$g(\varphi) = -\sum_1^\infty a_n(\gamma_n - \gamma)\cos n\varphi - \overline{U}(\cos\varphi - \gamma\sin\varphi) - \overline{V}(\sin\varphi + \gamma\cos\varphi) \qquad 2.16$$

It has been shown by Lauwerier in TW 43 that to the set of functions $\sin n\varphi + \gamma \cos n\varphi$ a set of functions $k_m(\varphi)$, $m=1,2,3\ldots$ is associated which have the property

$$\frac{\cos\mu\pi}{\pi}\int_0^\pi k_m(\varphi)(\sin n\varphi + \gamma\cos n\varphi)d\varphi = \delta_{m,n} \qquad 2.17$$

where
$$\mu = \frac{1}{\pi}\arctan\gamma \ .$$
An explicit representation of the $k_m(\varphi)$ is

$$k_m(\varphi) = 2\ \mathrm{ctn}^{2\mu}\tfrac{1}{2}\varphi\sum_1^m e_{m-j}\sin j\varphi\ , \qquad 2.18$$

where the coefficients e_n can be defined by means of a generating function, viz.

$$\left(\frac{1-t}{1+t}\right)^{2\mu} = \sum_0^\infty e_n t^n\ , \qquad |t| < 1\ . \qquad 2.19$$

Application of 2.17 to 2.16 yields

$$a_n = \frac{\cos\mu\pi}{\pi}\int_0^\pi k_n(\varphi)\ g(\varphi)d\varphi\ . \qquad 2.20$$

The evaluation of the integral in the right-hand side of 2.20 can be facilitated by making use of quantities $e_{m,n}$, defined by

$$\frac{\cos\mu\pi}{\pi}\int_0^\pi k_m(\varphi)(\sin n\varphi - \gamma\cos n\varphi)d\varphi = e_{m,n}\ . \qquad 2.21$$

The actual computation of the $e_{m,n}$ can be carried out by means of the recurrence relations, derived by Hofsommer (TW 48).
Application of 2.17 and 2.21 to 2.20 gives

$$a_n = \tfrac{1}{2}\sum_{m=1}^\infty a_m\left(\frac{1}{\theta_m}-1\right)(e_{m,n}-\delta_{m,n}) +$$
$$+\tfrac{1}{2}\left[\left(\gamma+\frac{1}{\gamma}\right)e_{m,1} + \left(\gamma-\frac{1}{\gamma}\right)\delta_{m,1}\right]\overline{U} + e_{m,1}\overline{V}\ . \qquad 2.22$$

This formula appears to be very suitable for an iterative procedure for the calculation of the coefficients a_n. In the following section we shall consider the case $\overline{V}=-1$, $\overline{U}=0$, which corresponds to a "Northern" wind that suddenly starts at t=0.

§ 3. Numerical calculations

We consider the case $\overline{U}=0$, $\overline{V}=-1$, $\Omega=1$, $\lambda=0.2$ for the two values of the Laplace variable $p=0.2$ and $p=0.3$. According to 2.9 and 2.10 we have

$$\bar{\zeta}(r,\varphi,p)=\sum_{1}^{\infty} a_n \frac{I_n(qr)}{q\,I_n'(q)} \sin n\varphi\ , \qquad 3.1$$

and

$$\sum_{1}^{\infty} a_n(\sin n\varphi + \gamma_n \cos n\varphi) = \sin\varphi + \gamma\cos\varphi\ . \qquad 3.2$$

A few values of q, γ and γ_n are

	$p = 0.2$	$p = 0.3$
q	0.762	0.866
γ	2.500	2.000
γ_1	2.195	1.696
γ_2	2.388	1.884
γ_3	2.443	1.940
γ_4	2.465	1.962
γ_5	2.475	1.974

We found the following values for the coefficients a_n

	$p = 0.2$	$p = 0.3$
a_1	1.052	1.080
a_2	-0.057	-0.080
a_3	0.033	0.043
a_4	-0.035	-0.048
a_5	0.029	0.039

The values of $\bar{\zeta}(r,\varphi,p_1)$ and $\bar{\zeta}(r,\varphi,p_2)$ are given below for $r=0\ (0.2)1$ and $\varphi=0(\frac{1}{8}\pi)\pi$

$\bar{\zeta}(r,\varphi,p_1)$

	r= .2	.4	.6	.8	1
$\varphi = \frac{\pi}{8}$	.065	.131	.197	.267	.341
$\frac{\pi}{4}$	.121	.242	.366	.495	.630
$\frac{3\pi}{8}$	.159	.319	.483	.655	.837
$\frac{\pi}{2}$	.172	.348	.528	.717	.920
$\frac{5\pi}{8}$	.160	.324	.494	.672	.858
$\frac{3\pi}{4}$	.123	.251	.385	.529	.684
$\frac{7\pi}{8}$	.067	.137	.213	.298	.397

$\bar{\zeta}(r,\varphi,p_2)$

$\varphi = \frac{\pi}{8}$	.063	.126	.191	.259	.332
$\frac{\pi}{4}$	.117	.235	.355	.481	.615
$\frac{3\pi}{8}$	.154	.309	.470	.640	.823
$\frac{\pi}{2}$	.168	.338	.516	.705	.910
$\frac{5\pi}{8}$	.156	.317	.485	.663	.853
$\frac{3\pi}{4}$	.120	.246	.381	.528	.690
$\frac{7\pi}{8}$	.065	.135	.212	.301	.409

By means of these values the influence of the exponential windfield 1.10 in the chosen points of the bay can be constructed. The maximum effect occurs at r=1, $\varphi = \frac{\pi}{2}$ where we have (see figure 1)

t = 0	$\zeta(1, \frac{\pi}{2}, t)$ = 0.74
3	1.23
6	1.95
9	2.86
12	3.48
15	2.10

From the tables giving the elevation at the other points of the semicircle a set of graphs has been constructed (figures 2,3,4,5,6,7,8) giving lines of equal disturbance at the various times.

4. The case $\Omega = 0$

If $\Omega = 0$ considerable simplifications occur. Since $\gamma = \infty$ formula 3.2 reduces to

$$\sum_{1}^{\infty} a_n \sin n\varphi = \sin \varphi \tag{4.1}$$

so that

$$\begin{cases} a_1 = 1 \\ a_n = 0 \quad , \quad \text{for} \quad n \geqq 2. \end{cases} \tag{4.2}$$

Hence 3.1 gives

$$\bar{\zeta}(r,\varphi,p) = \frac{I_1(qr)}{q\, I_1'(q)} \sin \varphi \quad . \tag{4.3}$$

In the numerical case with $\lambda = 0.2$ and $\Omega = 0$ we have $a_1 = 0.980$ for $p = 0.2$ and $a_1 = 0.964$ for $p = 0.3$.
The maximum effect of the exponential windfield 1.10 is again at $r=1$, $\varphi = \frac{\pi}{2}$ where we have (figure 1)

$t = 0$	$\zeta(1,\frac{\pi}{2},t) = 0.79$
3	1.31
6	2.09
9	3.06
12	3.75
15	2.34 .

In figure 2 the lines of equal disturbance at $t=12$ only are given. They are straight lines parallel to the ocean boundary.

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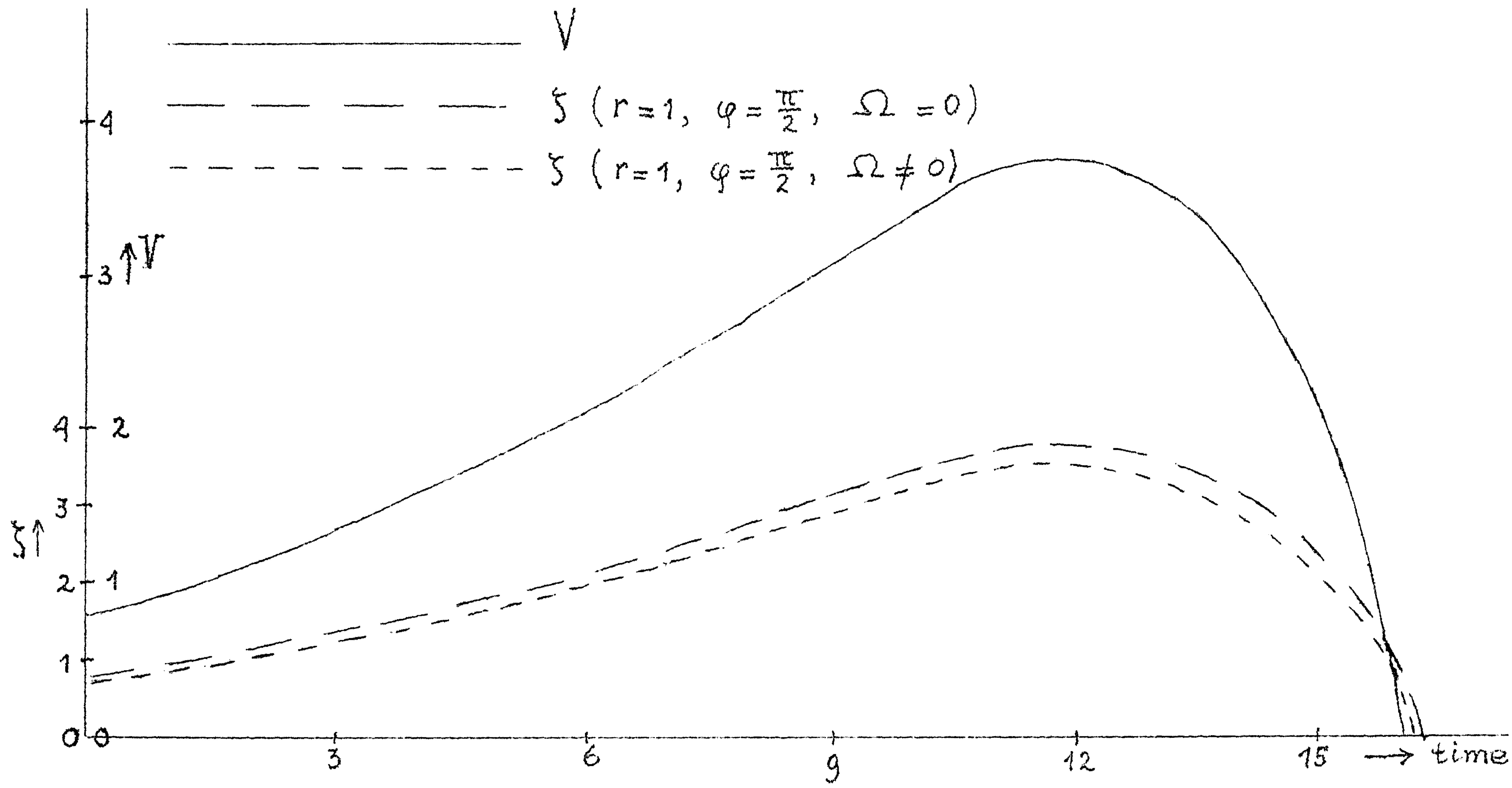

V
ζ (r = 1, φ = π/2, Ω = 0)
ζ (r = 1, φ = π/2, Ω ≠ 0)
ζ↑ V
0 3 6 9 12 15 → time

fig. 1

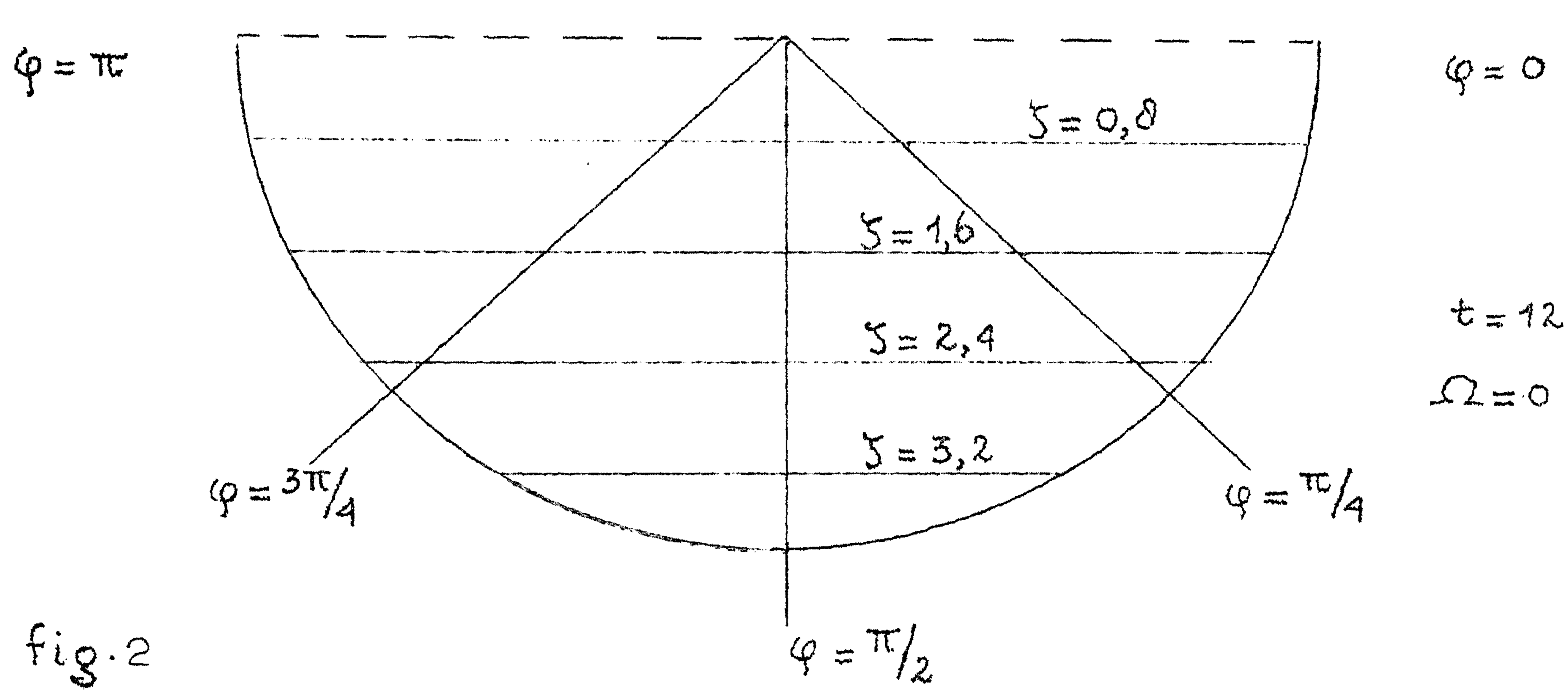
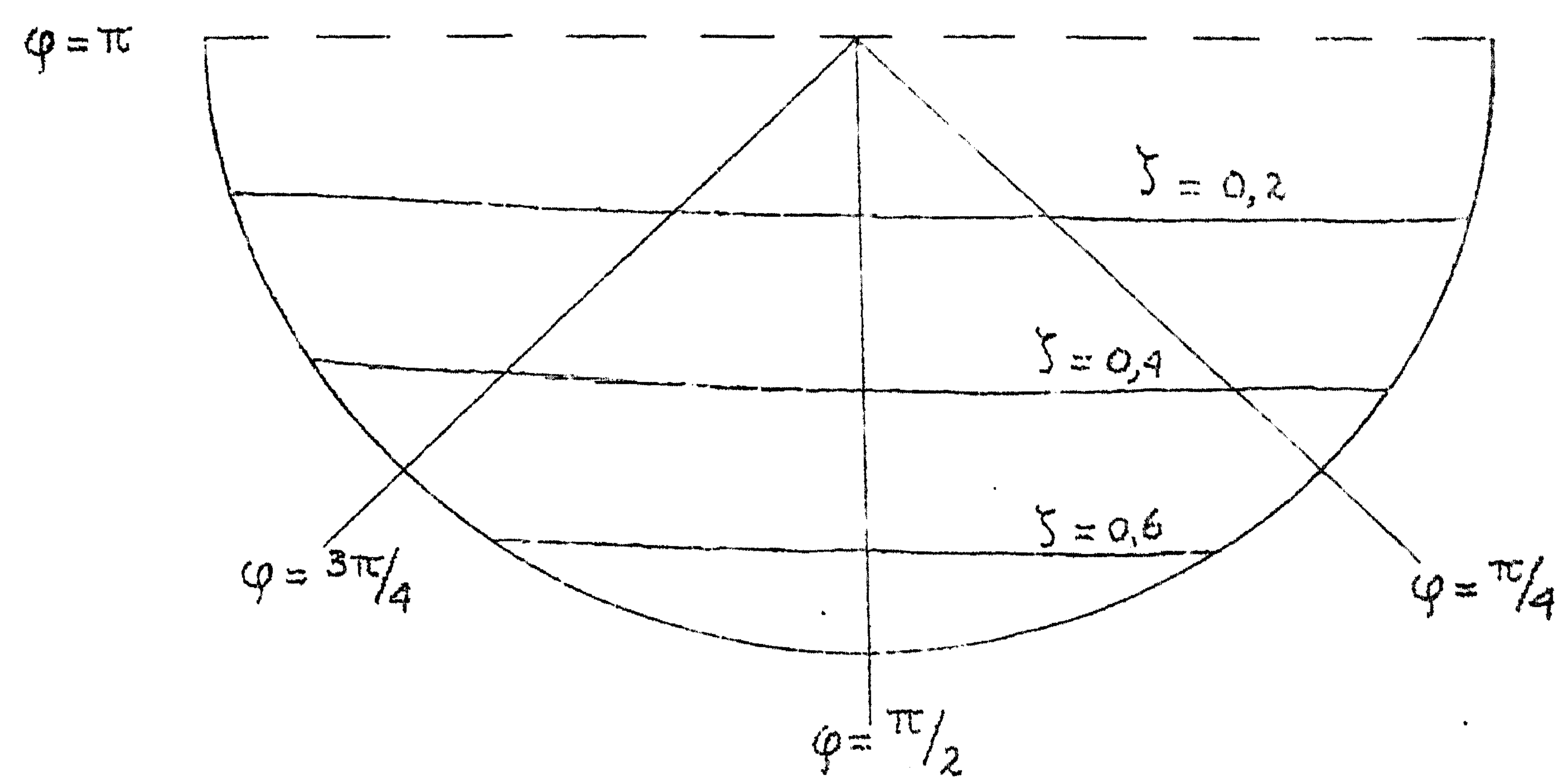

φ = π
φ = 0
ζ = 0,8
ζ = 1,6
ζ = 2,4
ζ = 3,2
t = 12
Ω = 0
φ = 3π/4
φ = π/4
φ = π/2

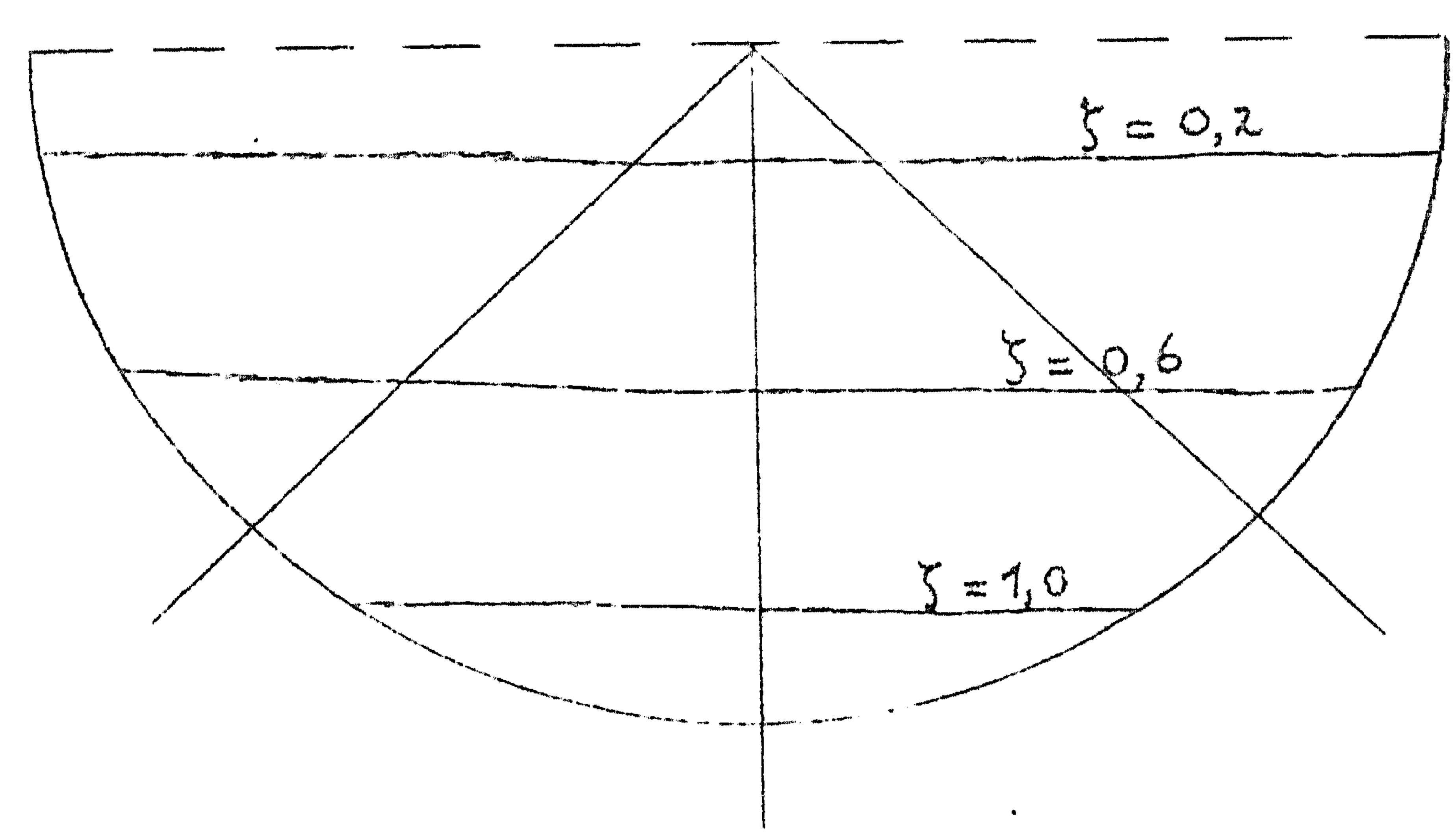
fig. 2

fig. 3



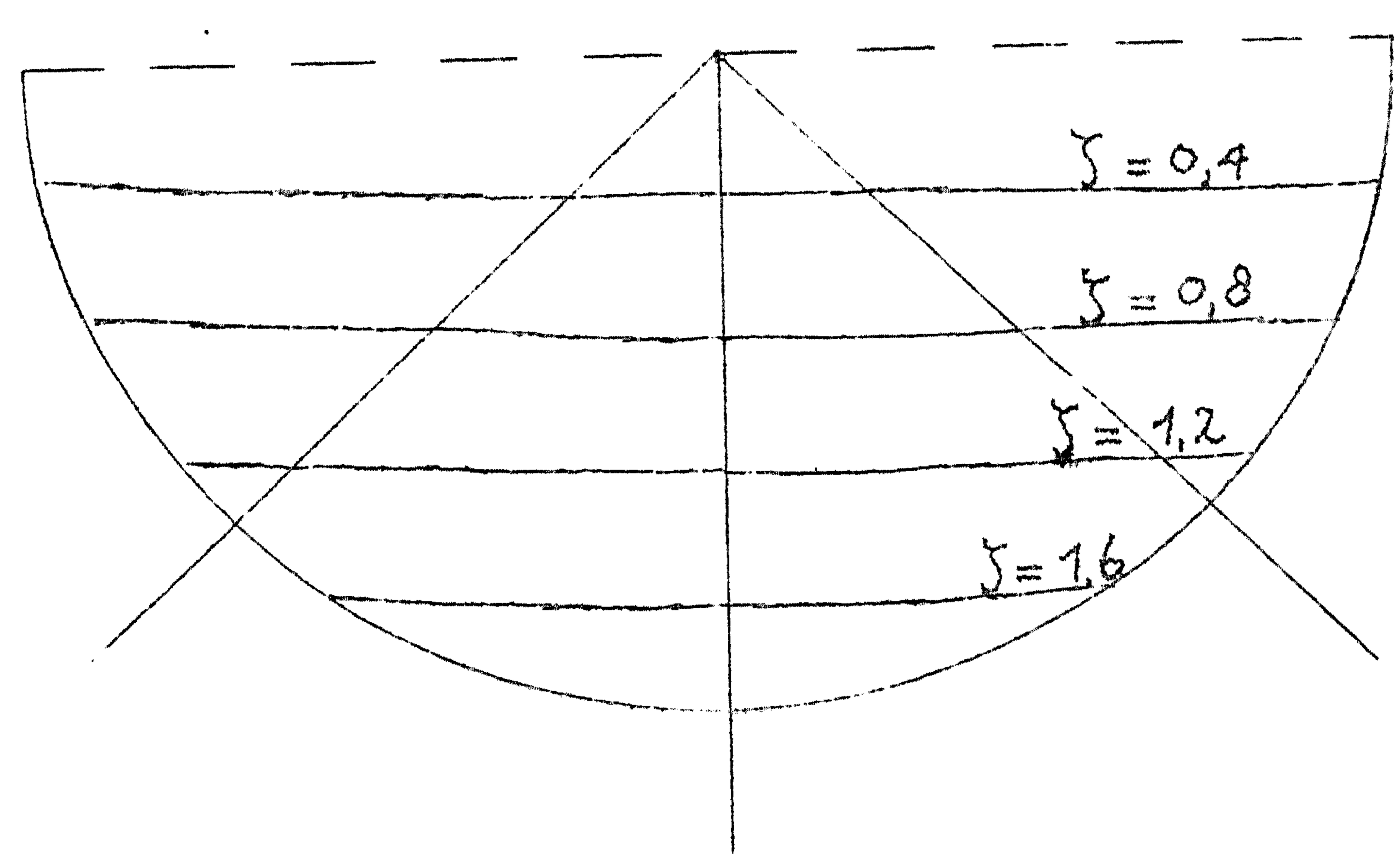
$t = 0$
$\Omega \neq 0$

fig. 4



$t = 3$
$\Omega \neq 0$

fig. 5



$t = 6$
$\Omega \neq 0$

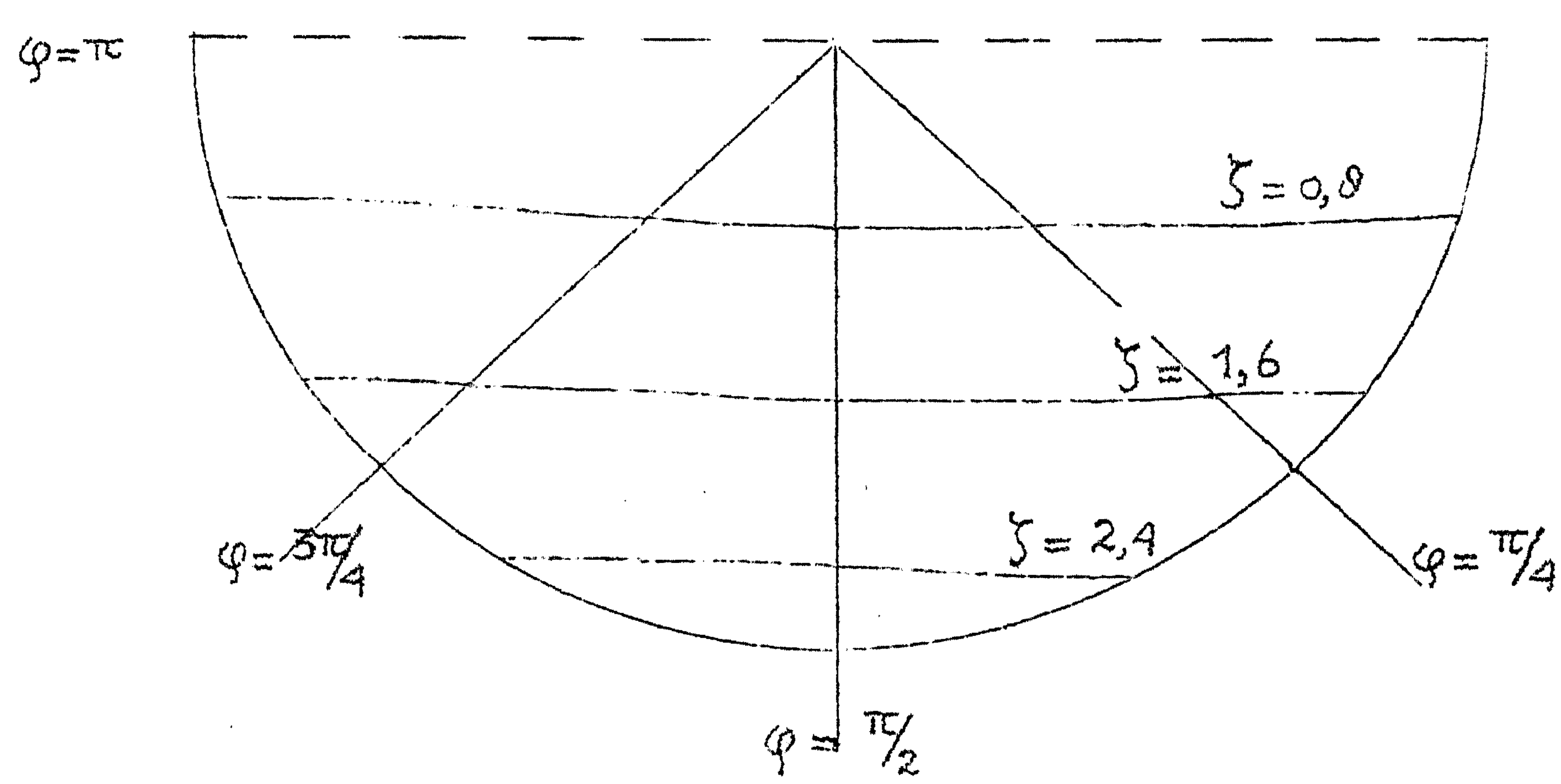


Fig. 6

$t = 9$

$\Omega \neq 0$

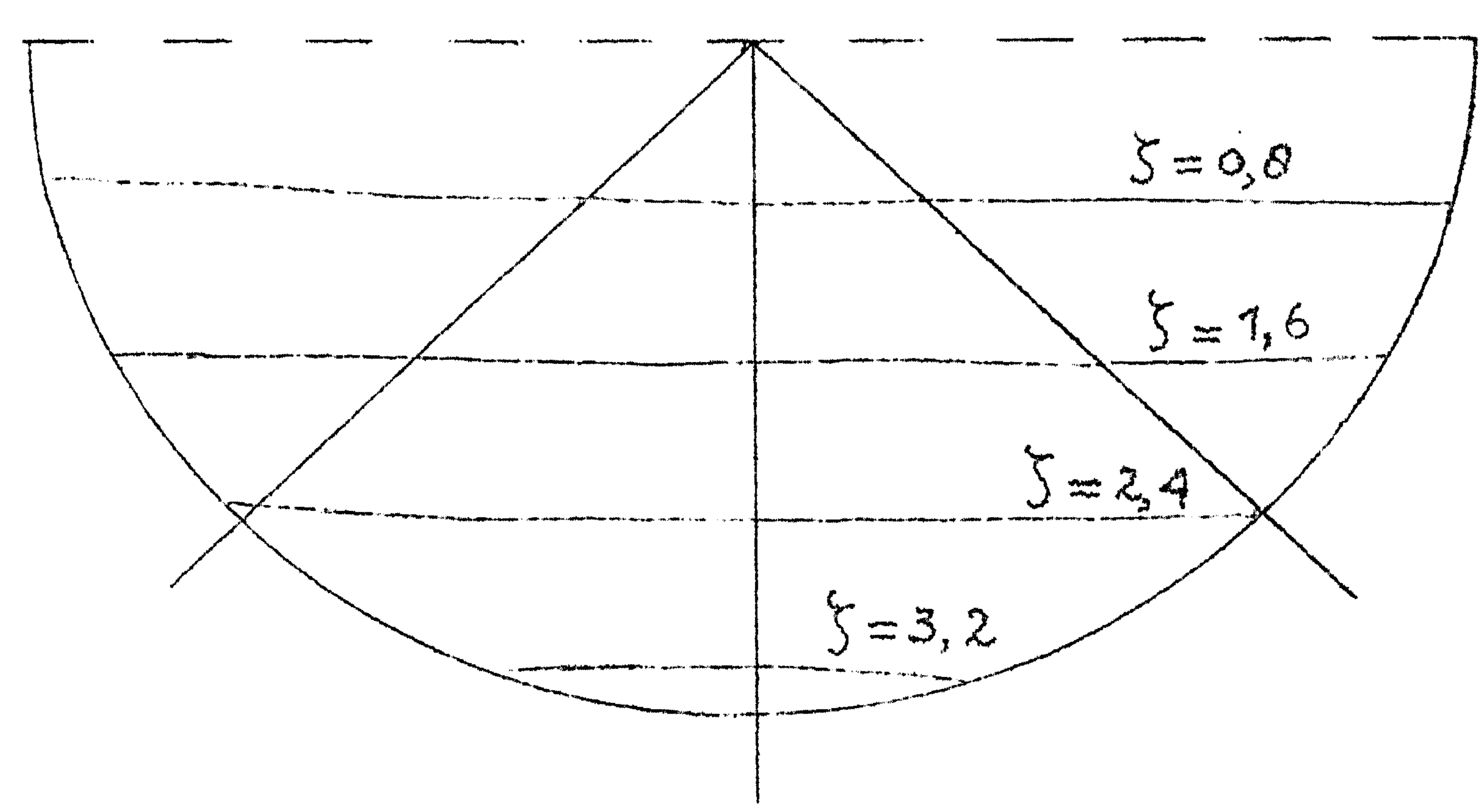


Fig. 7

$t = 12$

$\Omega \neq 0$

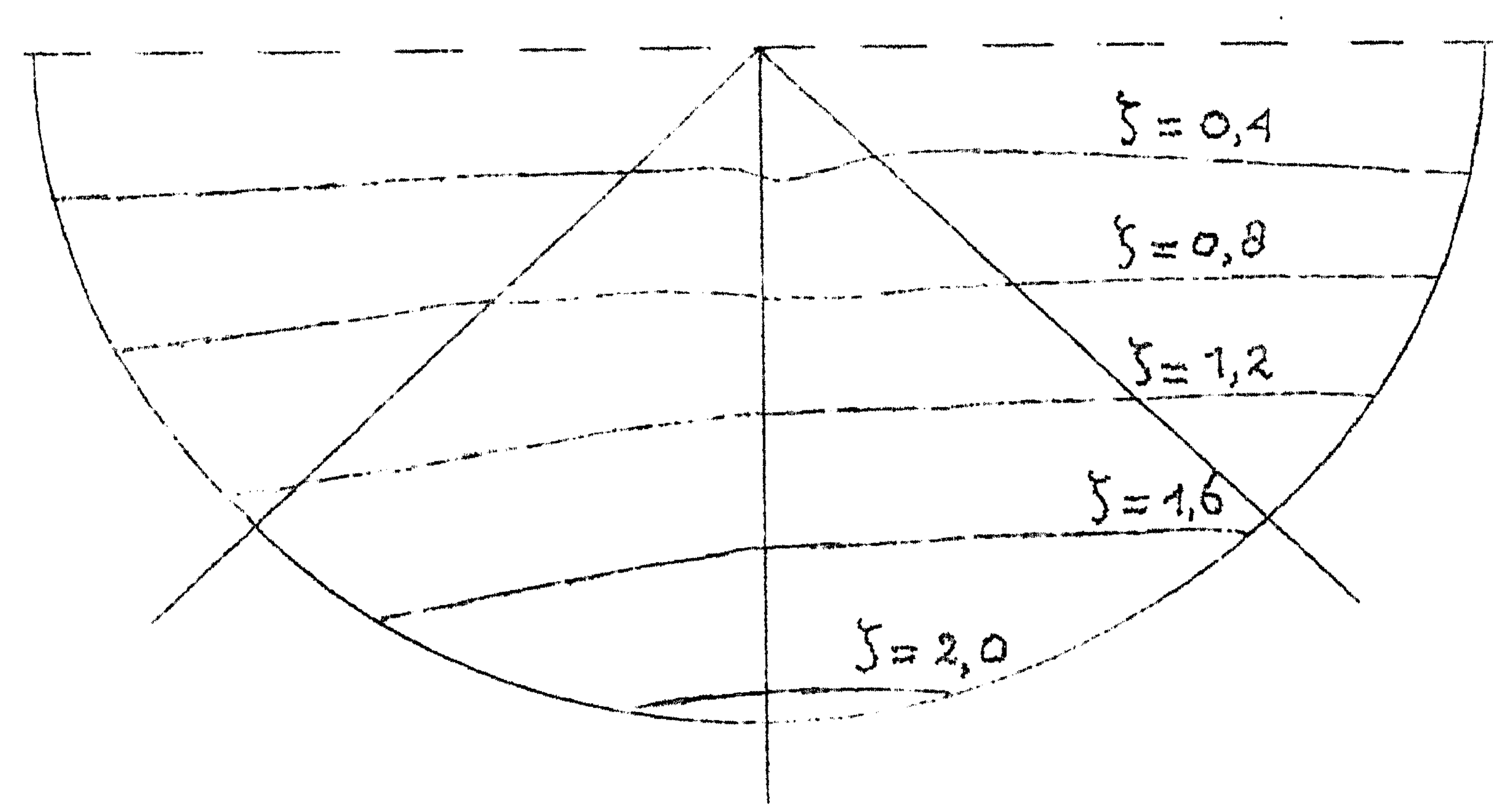


Fig. 8

$t = 15$

$\Omega \neq 0$